ISSN 2229-5518

On β-ρ-Continuity Where $\rho \in$ {L, M, R, S}

M. Priyadarshini, R. Selvi

Abstract— The authors Selvi.R, Thangavelu.P and Anitha.M introduced the concept of ρ-continuity between a topological space and a non empty set where $\rho \in$ {L, M, R, S} [4]. Navpreet singh Noorie and Rajni Bala[3] introduced the concept of f# function to characterize the closed, open and continuous functions. In this paper, the concept of β-ρ-continuity is introduced and its properties are investigated and β-ρ-continuity is further characterized by using f# functions.

KEYWORDS: Multifunction, Saturated set, β-ρ-continuity, β-continuity, β-open, β-closed and continuity.

1 INTRODUCTION:

By a multifunction F: X$\to$Y, We mean a point to set correspondence from X into Y with F(x)$\neq\phi$ for all x$\in$X. Any function f: X$\to$Y induces a multifunction $f^{-1}\circ f$: X$\to\wp$(X). It also induces another multifunction $f\circ f^{-1}$: Y$\to$(Y) provided f is surjective. The purpose of this paper is to introduce notions of β-L-Continuity, β- M-Continuity, β-R-Continuity and β-S-Continuity of a function f: X$\to$Y between a topological space and a non empty set. Here we discuss their links with β-open and β-closed sets. Also we establish pasting lemmas for β-R-continuous and β-S-continuous functions and obtain some characterizations for β-ρ-continuity. Navpreet singh Noorie and Rajni Bala [3] introduced the concept of $f^{\#}$ function to characterize the closed, open and continuous functions. The authors [6] characterized ρ-continuity by using $f^{\#}$ functions. In an analog way β-ρ-continuity is characterized in this paper.

2 PRELIMINARIES:

The following definitions and results that are due to the authors [4] and Navpreet singh Noorie and Rajni Bala [3] will be useful in sequel.

Definition: 2.1

Let f: (x, τ)$\to$Y be a function. Then f is

(i) L-Continuous if f^{-1}(f (A)) is open in X for every open set A in X. [4]

(ii) M-Continuous if f^{-1}(f (A)) is closed in X for every closed set A in X. [4]

Definition: 2.2

Let f: X$\to$(Y, σ) be a function. Then f is

(i) R-Continuous if f (f -1(B)) is open in Y for every open set B in Y. [4]

(ii) S-Continuous if f (f -1(B)) is closed in Y for every closed set B in Y. [4]

Definition 2.3:

Let f: X$\to$Y be any map and E be any subset of X. then the following hold.

(i) $f^{\#}$(E) = {y $\in$ Y: f^{-1}(y)$\subseteq$E} ; (ii) $E^{\#}$ = f^{-1}($f^{\#}$(E)). [3]

Lemma 2.4:

Let E be a subset of X and let f: X$\to$Y be a function. Then the following hold.

(i) $f^{\#}$(E) = Y\f(X\E) ; (ii) f(E) = Y\$f^{\#}$(X\E). [3]

Lemma 2.5:

Let E be a subset of X and let f: X$\to$Y be a function. Then the following hold.

(i) f^{-1}($f^{\#}$(E)) = X\f^{-1}(f(X\E)); (ii) f^{-1}(f(E)) = X\ f^{-1}($f^{\#}$(X\E)) . [6]

Lemma 2.6:

Let E be a subset of X and let f: X$\to$Y be a function. Then the following hold.

(i) $f^{\#}$(f^{-1}(E)) = Y\f (f^{-1}(Y\E)) ; (ii) f(f^{-1}(E)) = Y\$f^{\#}$ (f^{-1}(Y\E)) . [6]

Definition 2.7:

Let f: X$\to$Y, A$\subseteq$X and B$\subseteq$Y. we say that A is f-saturated if f^{-1}(f(A))$\subseteq$A and B is f^{-1}-saturated if f(f^{-1}(B)) $\subseteq$B. Equivalently A is f-saturated if and only if f^{-1}(f (A))=A, and B is f^{-1}-saturated if and only if f (f^{-1}(B))=B.

Definition 2.8:

Let A be a subset of a topological space (X, τ). Then A is called

(i) Semi-open if A$\subseteq$cl(int(A)) and semi-closed if int(cl(A)) $\subseteq$ A; [1].

(ii) Pre-open if A$\subseteq$int(cl(A)) and pre-closed if cl(int(A))$\subseteq$ A; [2].

(iii) α -open if A$\subseteq$int(cl(int(A))) and α -closed if cl(int(cl(A))) $\subseteq$ A; [7].

(iv) semi-pre-open or β-open if A$\subseteq$cl(int(cl(A))) and semi-pre-closed or β-closed if int(cl(int(A)))$\subseteq$A; [8].

Definition 2.9:

Let f: $(X,\tau)\to(Y,\sigma)$ be a function. Then f is β-continuous if $f^{-1}(B)$ is open in X for every β-open set B in Y. [8]

Definition: 2.10:

Let f: $(X,\tau)\to(Y,\sigma)$ be a function. Then f is β-open (resp. β-closed) if f(A) is β-open(resp. β-closed) in Y for every β-open(resp. β -closed) set A in X.

3. β-ρ-CONTINUITY WHERE $\rho \in$ {L, M, R, S}

Definition: 3.1

Let f: $(X,\tau)\to$Y be a function. Then f is

(i) β-L-Continuous if $f^{-1}(f(A))$ is open in X for every β-open set A in X.

(ii) β-M-Continuous if $f^{-1}(f(A))$ is closed in X for every β-closed set A in X.

Definition: 3.2

Let f: X $\to(Y,\sigma)$ be a function. Then f is

(i) β-R-Continuous if $f(f^{-1}(B))$ is open in Y for every β-open set B in Y.

(ii) β -S-Continuous if $f(f^{-1}(B))$ is closed in Y for every β-closed set B in Y.

Example: 3.3

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let τ = { Φ, X ,{a}, {b}, {a, b}, {a, b, c} }. Let f: $(X,\tau)\to$Y defined by f(a)=1, f(b)=2, f(c)=3, f(d)=4. Then f is β-L-Continuous and β-M-Continuous.

Example: 3.4

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let σ ={ Φ ,Y, {1},{2},{1,2},{1,2,3} }. Let g : X$\to(Y,\sigma)$ defined by g(a)=1, g(b)=2, g(c)=3, g(d)=4. Then g is β-R-Continuous and β-S-Continuous.

Definition: 3.5

Let f: $(X,\tau)\to(Y,\sigma)$ be a function, Then f is

(i) β-LR-Continuous, if it is both β-L-Continuous and β-R-Continuous.

(ii) β-LS -Continuous, if it is both β-L-Continuous and β-S-Continuous.

(iii) β-MR-Continuous, if it is both β-M-Continuous and β-R-Continuous.

(iv) β-MS-Continuous, if it is both β-M-Continuous and β-S-Continuous.

Theorem: 3.6

(i) Every injective function f:$(X,\tau)\to(Y,\sigma)$ is β-L-Continuous and β-M-Continuous.

(ii) Every surjective function f:$(X,\tau)\to(Y,\sigma)$ is β-R-Continuous and β-S-Continuous.

(iii) Any constant function f:$(X,\tau)\to(Y,\sigma)$ is β-R-Continuous and β-S-Continuous.

Proof:

(i) Let f: $(X,\tau)\to(Y,\sigma)$ be injective function. Then β-L-Continuity and β-M-Continuity follow from the fact that $f^{-1}(f(A))$=A. This proves (i).

(ii) Let f: $(X,\tau)\to(Y,\sigma)$ be surjective function. Since f is surjective, $f(f^{-1}(B))$=B for every subset B of Y. Then f is both β-R-Continuous and β-S-Continuous. This proves (ii).

(iii)Suppose f(x) =y_o for every x in X. Then $f(f^{-1}(B))$ =Y if $y_o \in B$ and $f(f^{-1}(B)) = \Phi$, if $y_o \in Y\backslash B$. This proves (iii).

Corollary: 3.7

If f: $(X,\tau)\to(Y,\sigma)$ be bijective function then f is β-L-Continuous, β -M-Continuous, β-R-Continuous and β-S-Continuous .

Theorem: 3.8

Let f: $(X,\tau)\to(Y,\sigma)$.

(i) If f is L-Continuous (resp. M-Continuous) then it is β-L-Continuous (resp. β-M-Continuous).

(ii) If f is R-Continuous (resp. S-Continuous) then it is β-R-Continuous (resp. β-S-Continuous).

Proof:

(i) Let A$\subseteq$X be β-open (resp. β-closed) in X. Since every β-open (resp. β-closed) set is open (resp. closed) and since f is L-continuous (resp. M-continuous) $\Rightarrow f^{-1}(f(A))$ is open (resp. closed) in X. Therefore f is β-L-Continuous (resp. β -M-Continuous).

(ii) Let B$\subseteq$Y be β-open (resp. β-closed) in Y. since every β-open (resp. β-closed) set is open (resp. closed) and since f is R-continuous (resp. S-continuous) $\Rightarrow f(f^{-1}(B))$ is open (resp. closed) in Y. Therefore f is β-R-Continuous (resp. β-S-Continuous).

Theorem: 3.9

Let f: $(X,\tau)\to$Y be β-L-Continuous. Then cl(int(cl(A))) is f-saturated whenever A is f-saturated and α -closed.

Proof:

Let A$\subseteq$X be f-saturated. Since f is β-L-Continuous$\Rightarrow$A is β-open set in X$\Rightarrow$A$\subseteq$ cl(int(cl(A))). And since A is α -closed$\Rightarrow$ cl(int(cl(A))) $\subseteq$A. Therefore cl(int(cl(A)))= A. Since A is f-saturated$\Rightarrow$ $f^{-1}(f(A))$ = A. That implies cl(int(cl(A)))=f^{-1}(f(cl(int(cl(A))))). Therefore Hence cl(int(cl(A))) is f-saturated whenever A is f-saturated and α -closed.

ISSN 2229-5518

Corollary: 3.10

Let f: $(X,\tau) \to Y$ be β-L-Continuous. Then $cl(int(cl(f^{-1}(B))))$ is f-saturated for every subset B of Y.

Proof:

Let $B \subseteq Y$. we know that $f(f^{-1}(B)) \subseteq B$, Then $f^{-1}(f(f^{-1}(B))) \subseteq f^{-1}(B)$. Also $f^{-1}(B) \subseteq f^{-1}(f(f^{-1}(B))) \subseteq f^{-1}(B)$. So that $f^{-1}(f(f^{-1}(B))) = f^{-1}(B)$. This proves that $f^{-1}(B)$ is f-saturated, and hence by using theorem: 3.9, $cl(int(cl(f^{-1}(B))))$ is f-saturated.

Theorem: 3.11

Let f: $(X,\tau) \to Y$ be β-M-Continuous. Then $int(cl(int(A)))$ is f-saturated whenever A is f-saturated and α-open.

Proof:

Let $A \subseteq X$ be f-saturated. Since f is β-M-Continuous (i)
$\Rightarrow$ A is β-closed set in X $\Rightarrow int(cl(int(A))) \subseteq A$ and since A is α-open $\Rightarrow A \subseteq int(cl(int(A)))$. (ii)
Therefore $int(cl(int(A)))$ =A. Since A is f-saturated $\Rightarrow f^{-1}(f(A))= A$.
That implies $int(cl(int(A))) = f^{-1}(f(int(cl(int(A)))))$. Hence $int(cl(int(A)))$ is f-saturated whenever A is f-saturated and α-open.

Theorem: 3.12

Let f: $X \to (Y, \sigma)$ be β-R-Continuous. Then $cl(int(cl(B)))$ is f^{-1} – saturated whenever B is f^{-1} –saturated and α-closed.

Proof:

Let $B \subseteq Y$ be f^{-1} –saturated.
Since f is β-R-Continuous $\Rightarrow$ B is β-open set in Y,
$\Rightarrow cl(int(cl(B))) \supseteq B$, and since B is β-closed
$\Rightarrow cl(int(cl(B))) \subseteq B$, Therefore $cl(int(cl(B)))$=B, (i)
since B is f^{-1} –saturated $\Rightarrow f(f^{-1}(B)) = B$,
which implies that $f(f^{-1}(cl(int(cl(B)))))=cl(int(cl(B)))$, (ii)
Therefore hence $cl(int(cl(B)))$ is f^{-1}-saturated whenever B is f^{-1} –saturated and α-closed.

Theorem: 3.13

Let f: $X \to (Y,\sigma)$ be β-S-Continuous Then $int(cl(int(B)))$ is f^{-1}– saturated whenever B is f^{-1}–saturated and α-open .

Proof:

Let $B \subseteq Y$ be f^{-1}–saturated. Since f is β-S-Continuous $\Rightarrow$ B is β-closed set in Y $\Rightarrow int(cl(int(B))) \subseteq B$ and since B is α-open $\Rightarrow int(cl(int(B))) \supseteq B$,
Therefore $int(cl(int(B)))$=B, since B is f^{-1} –saturated, $f(f^{-1}(B)) = B$.
Which implies that $int(cl(int(B)))=f(f^{-1}(int(cl(int(B)))))$,
Therefore hence $int(cl(int(B)))$is f^{-1}-saturated whenever B is f^{-1} –saturated and α-open.

Corollary: 3.14

Let f: $(X,\tau) \to (Y,\sigma)$ be β-S-Continuous Then $int(cl(int(f(A))))$ is f^{-1} – saturated for every subset A of X.

Proof:

Let $A \subseteq X$. We know that $f^{-1}(f(A)) \supseteq A$, Then $f(f^{-1}(f(A))) \supseteq f(A)$, Also $f(A) \supseteq f(f^{-1}(f(A))) \supseteq f(A)$, So that $f(f^{-1}(f(A))) = f(A)$. This proves that hence by using (theorem 3.13) $int(cl(int(f(A))))$ is f^{-1} – saturated.

4 Properties

In this section we prove certain theorems related with β-open and β-closed functions.

Theorem: 4.1

Let f: $(X,\tau) \to (Y,\sigma)$ be β-open and β-Continuous, Then f is β-L-Continuous.

Let f: $(X,\tau) \to (Y,\sigma)$ be open and β-Continuous, Then f is β-R-Continuous.

Proof:

(i) Let $A \subseteq X$ be β-open in X. Let f: $(X,\tau) \to (Y,\sigma)$ be β-open and β-Continuous, since f is β-open,
$\Rightarrow$ f(A) is β-open in Y and since f is β-continuous,
$\Rightarrow f^{-1}(f(A))$ is open in X. Therefore f is β-L Continuous,
This proves (i).
(ii) Let $B \subseteq Y$ be β-open in Y. Let f: $(X,\tau) \to (Y,\sigma)$ be open and β- Continuous, since f is β-continuous $\Rightarrow f^{-1}(B)$ is open in X, and since f is open $\Rightarrow f(f^{-1}(B))$ is open in Y, Therefore f is β-R Continuous, This proves (ii) .

Theorem: 4.2

Let f: $(X,\tau) \to (Y,\sigma)$ be β-closed and β-Continuous, Then f is β-M-Continuous.

Let f: $(X,\tau) \to (Y,\sigma)$ be closed and β-Continuous, Then f is β-S-Continuous.

Proof:

(i) Let $A \subseteq X$ be β-closed in X. Let f: $(X,\tau) \to (Y,\sigma)$ be β-closed and β-Continuous, since f is β-closed $\Rightarrow$ f(A) is β-closed in Y and since f is β-continuous $\Rightarrow f^{-1}(f(A))$ is closed in X. Therefore f is β-M Continuous. This proves (i).
(ii) Let $B \subseteq Y$ be β-closed in Y. Let f: $(X,\tau) \to (Y,\sigma)$ be closed and β-Continuous, since f is β-continuous $\Rightarrow f^{-1}(B)$ is closed in X and since f is closed $\Rightarrow f(f^{-1}(B))$ is closed in Y. Therefore f is β-S Continuous, This proves (ii).

ISSN 2229-5518

Theorem: 4.3

Let X be a topological space.

If A is an β-open subspace of X, the inclusion function $j: A \to X$ is β-L-continuous and β-R-continuous.

If A is an β-closed subspace of X, the inclusion function $j: A \to X$ is β-M-continuous and β-S-continuous.

Proof:

(i) Suppose A is an β-open subspace of X. Let $j: A \to X$ be an inclusion function. Let $U \subset X$ be β-open in X then $j(j^{-1}(U)) = j(U \cap A) = U \cap A$ Which is open in X. Hence j is β-R-continuous. Now, let $U \subseteq A$ be β-open in A. Then $j^{-1}(j(U)) = j^{-1}(U) = U$ which is open in A. Hence j is β-L-continuous, this proves (i).

(ii) Suppose A is an β-closed subspace of X. Let $j: A \to X$ be an inclusion function. Let $U \subset X$ be β-closed in X then $j(j^{-1}(U)) = j(U \cap A) = U \cap A$, Which is closed in X. Hence j is β-S-continuous. Now, let $U \subseteq A$ be β-closed in A. Then $j^{-1}(j(U)) = j^{-1}(U) = U$ which is closed in A. Hence j is β-M-continuous, this proves (ii).

Theorem: 4.4

Let $g: Y \to Z$ and $f: X \to Y$ be any two functions. Then the following hold.

(i) If $g: Y \to Z$ is β-L-continuous (resp. β-M-continuous) and $f: X \to Y$ is β-open (resp. β-closed) and continuous, then $g \circ f: X \to Z$ is β-L-continuous (resp. β-M-continuous)

(ii) If $g: Y \to Z$ is open (resp. closed) and β-continuous and $f: X \to Y$ is R-continuous (resp. S-continuous), then $g \circ f$ is β-R-continuous (resp. β-S-continuous).

Proof:

(i) Suppose g is β-L-continuous (resp. β-M continuous) and f is β-open (resp. β-closed) and continuous. Let A be β-open (resp. β-closed) in X.

Then $(g \circ f)^{-1}.(g \circ f)(A) = f^{-1}(g^{-1}(g(f(A))))$. Since f is β-open (resp. β-closed) $\Rightarrow$ f (A) is β-open (resp. β-closed) in Y.

Since g is β-L-continuous (resp. β-M-continuous) $\Rightarrow g^{-1}(g(f(A)))$is open (resp. closed) in Y. Since f is continuous $\Rightarrow f^{-1}(g^{-1}(g(f(A))))$ is open (resp. β-closed) in X. Therefore, $g \circ f$ is β-L-continuous (resp. β-M-continuous). This proves (i).

(ii) Let $f: X \to Y$ be R-continuous (resp. S-continuous) and $g: Y \to Z$ be open (resp. closed) and β-continuous.

Let B be β-open (resp. β-closed) in Z. Then $(g \circ f)(g \circ f)^{-1}(B) = (g \circ f)(f^{-1}(g^{-1}(B))) = g(f(f^{-1}(g^{-1}(B))))$.

Since g is β-continuous $g^{-1}(B)$ is open (resp. closed) in Y.

Since f is R-continuous (resp. S-continuous) $\Rightarrow f(f^{-1}(g^{-1}(B)))$ is open (resp. closed) in Y.

Since g is open (resp. closed) $\Rightarrow g(f(f^{-1}(g^{-1}(B))))$ is open (resp. closed) in Z. Therefore, $g \circ f$ is β-R-continuous (resp. β-S-continuous). This proves (ii).

Theorem: 4.5

If $f: X \to Y$ is β-L-continuous and if A is an open subspace of X, then the restriction of f to A is β-L-continuous.

Proof:

Let $h = f/_A$. Then $h = f \circ j$, where j is the inclusion map $j: A \to X$ since j is open and continuous and since $f: X \to Y$ is β-L-continuous, using theorem (4.4 (i)) $\Rightarrow$ h is β-L-continuous.

Theorem: 4.6

If $f: X \to Y$ is β-M-continuous and if A is a closed subspace of X, then the restriction of f to A is β-M-continuous.

Proof:

Let $h = f/_A$. Then $h = f \circ j$, where j is the inclusion map $j: A \to X$ since j is closed and continuous and since $f: X \to Y$ is β-M-continuous, using theorem (4.4 (i)),

$\Rightarrow$ h is β-M-continuous.

Theorem: 4.7

Let $f: X \to Y$ be β-R-continuous. Let $f(x) \subseteq Z \subseteq Y$ and f(X) be open in Z. Let $h: X \to Z$ be obtained by from f by restricting the co-domain of f to Z. Then h is β-R-continuous.

Proof:

Clearly $h = j \circ f$ where $j: f(x) \to Z$ is an inclusion map. Since f(X) is open in Z, the inclusion map j is both open and β-continuous. Then by applying theorem 4.4(ii),

$\Rightarrow$ h is β-R-continuous.

Theorem: 4.8

Let $f: X \to Y$ be β-S-continuous. Let $f(x) \subseteq Z \subseteq Y$ and f(X) be closed in Z. Let $h: X \to Z$ be obtained by from f by restricting the co-domain of f to Z. Then h is β-S-continuous.

Proof:

Clearly $h = j \circ f$ where $j: f(x) \to Z$ is an inclusion map. Since f(X) is closed in Z, the inclusion map j is both closed and β-continuous. Then by applying theorem 4.4(ii)

$\Rightarrow$ h is β-S-continuous.

Now we establish the pasting lemmas for β-R-continuous and β-S-continuous functions.

International Journal of Scientific & Engineering Research, Volume 6, Issue 5, May-2015
ISSN 2229-5518

Theorem: 4.9

Let $X=A \bigcup B$. Let $f: A \to (Y, \sigma)$ and $g: B \to (Y, \sigma)$ be β-R-continuous (res. β-S-continuous) $f(x)= g(x)$ for every $x \in A \bigcap B$, then f and g combined to give a β-R-continuous (res. β-S-continuous) function $h: X \to Y$ defined by $h(x)=f(x)$ if $x \in A$, and $h(x)=g(x)$ if $x \in B$.

Proof:

Let C be a β-open (res. β-closed) set in Y.
Now $hh^{-1}(C) = h(f^{-1}(C) \bigcup g^{-1}(C)) = h(f^{-1}(C)) \bigcup h(g^{-1}(C)) = f(f^{-1}(C)) \bigcup g(g^{-1}(C))$. Since f is β-R-continuous (res. β-S-continuous), $f(f^{-1}(C))$ is open (resp. closed) in Y and Since g is β-R-continuous (res. β-S-continuous), $g(g^{-1}(C))$ is open (resp. closed) in Y. Therefore, $hh^{-1}(C)$ is open (resp. closed) in Y. Hence h is β-R-continuous (resp. β-S-continuous).

CHARACTERIZATIONS

Theorem: 5.1

A function $f: X \to Y$ is β-L-continuous if and only if $f^{-1}(f^{\#}(A))$ is closed in X for every β-closed subset A of X .

Proof:

Suppose f is β-L-continuous. Let A be β-closed in X. Then $G = X / A$ is β-open in X. Since f is β-L-continuous and since G is β-open in $X \Rightarrow f^{-1}(f(G))$ is open in X.
By applying lemma ((2.5)-(i)) $\Rightarrow f^{-1}(f^{\#}(A)) = X \setminus f^{-1}(f(X \setminus A)) = X \setminus f^{-1}(f(G))$. That implies $f^{-1}(f^{\#}(A))$ is closed in X.
Conversely, we assume that $f^{-1}(f^{\#}(A))$ is closed in X for every β-closed subset A of X. Let G be a β-open in X.
By our assumption, $f^{-1}(f^{\#}(A))$ is closed in X, where $A = X \setminus G$. By using lemma ((2.5)-(ii)) $\Rightarrow f^{-1}(f(G)) = X \setminus f^{-1}(f^{\#}(X \setminus G)) = X \setminus f^{-1}(f^{\#}(A))$.
That implies $f^{-1}(f(G))$ is open in X. Therefore, hence f is β-L-continuous.

Theorem: 5.2

A function $f: X \to Y$ is β-M-continuous if and only if $f^{-1}(f^{\#}(G))$ is open in X for every β-open subset G of X.

Proof:

Suppose f is β-M-continuous. Let G be β-open in X. Then $A = X \setminus G$ is β-closed in X. Since f is β-M-continuous and since A is β-closed in $X \Rightarrow f^{-1}(f(A))$ is closed in X.
By applying lemma ((2.5)-(i)) $\Rightarrow f^{-1}(f^{\#}(G)) = X \setminus f^{-1}(f(X \setminus G)) = X \setminus f^{-1}(f(A))$. That implies $f^{-1}(f^{\#}(G))$ is open in X.
Conversely, we assume that $f^{-1}(f^{\#}(G))$ is open in X for every β-open subset G of X. Let A be a β-closed in X.
By our assumption, $f^{-1}(f^{\#}(G))$ is open in X, where $G = X \setminus A$.
By using lemma ((2.5)-(ii)) $\Rightarrow f^{-1}(f(A)) = X \setminus f^{-1}(f^{\#}(X \setminus A)) = X \setminus f^{-1}(f^{\#}(G))$. That implies $f^{-1}(f(A))$ is open in X.
Therefore, hence f is β-M-continuous.

Theorem: 5.3

A function $f: X \to Y$ is β-R-continuous if and only if $f^{\#}(f^{-1}(B))$ is closed in Y for every β-closed subset B of Y.

Proof:

Suppose f is β-R-continuous. Let B be β-closed in Y. Then $G=Y \setminus B$ is β-open in Y. Since f is β-R-continuous and since G is β-open in $Y \Rightarrow f(f^{-1}(G))$ is open in Y.
Now by using lemma ((2.6)(i)) $\Rightarrow f^{\#}(f^{-1}(B)) = Y \setminus f(f^{-1}(Y \setminus B)) = Y \setminus f(f^{-1}(G))$. That implies $f^{\#}(f^{-1}(B))$ is closed in Y.
Conversely, we assume that $f^{\#}(f^{-1}(B))$ is closed in Y for every β-closed subset B of Y. Let G be β-open in Y.
Let $B = Y \setminus G$. By our assumption, $f^{\#}(f^{-1}(B))$ is closed in Y.
By lemma ((2.6)(ii))
$\Rightarrow f(f^{-1}(G))=Y \setminus (f^{\#}(f^{-1}(Y \setminus G)))=Y \setminus f^{\#}(f^{-1}(B))$,
This proves that $f(f^{-1}(G))$ is open in Y. Therefore, hence f is β-R-continuous.

Theorem: 5.4

The function $f: X \to Y$ is β-S-continuous if and only if $f^{\#}(f^{-1}(G))$ is open in Y for every β-open subset G of Y.

Proof:

Suppose f is β-S-continuous. Let G be β-open in Y. Then $B=Y \setminus G$ is β-closed in Y. Since f is β-S-continuous and since B is β-closed in $Y \Rightarrow f(f^{-1}(B))$ is open in Y.
Now by using lemma ((2.6)(i)) $\Rightarrow f^{\#}(f^{-1}(G)) = Y \setminus f(f^{-1}(Y \setminus G)) = Y \setminus f(f^{-1}(B))$. That implies $f^{\#}(f^{-1}(G))$ is open in Y.
Conversely, we assume that $f^{\#}(f^{-1}(G))$ is open in Y for every β-open subset G of Y. Let B be β-closed in Y.
Let $G = Y \setminus B$. By our assumption, $f^{\#}(f^{-1}(G))$ is open in Y.
By lemma ((2.6)(ii))
$\Rightarrow f(f^{-1}(B))=Y \setminus (f^{\#}(f^{-1}(Y \setminus B)))=Y \setminus f^{\#}(f^{-1}(G))$,
This proves that $f(f^{-1}(B))$ is closed in Y. Therefore, hence f is β-S-continuous.

Theorem: 5.5

Let $f: (X, \tau) \to Y$ be a function . Then the following are equivalent.
(i) f is β-L-continuous,
(ii) for every β-closed subset A of X, $f^{-1}(f^{\#}(A)$ is closed in X,
(iii)for every $x \in X$ and for every β-open set U in X with $f(x) \in f(U)$ there is an open set G in X with $x \in G$ and $f(G) \subseteq f(U)$,
(iv) $f^{-1}(f(cl(int(cl(A))))) \subseteq int(f^{-1}(f(A)))$ for every α-closed subset A of X.
(v) $cl(f^{-1}(f^{\#}(A))) \subseteq f^{-1}(f^{\#}(int(cl(int(A)))))$ for every α-open subset A of X.

Proof:

(i) $\Leftrightarrow$ (ii): follows from theorem 5.1.
(i) $\Leftrightarrow$ (iii): Suppose f is β-L-continuous.

ISSN 2229-5518

Let U be β-open set in X such that $f(x) \in f(U)$.

Since f is β-L-continuous, $f^{-1}(f(U))$ is open in X.

Since $x \in f^{-1}(f(U))$ there is an open set G in X such that $x \in G \subseteq f^{-1}(f(U)) \Rightarrow f(G) - f(f^{-1}(f(U))) \subseteq f(U)$.

This proves (iii).

Conversely, suppose (iii) holds. Let U be β-open set in X and $x \in f^{-1}(f(U))$. Then $f(x) \in f(U)$. By using (iii), there is an open set G in X containing x such that $f(G) \subseteq f(U)$.

Therefore $x \in G \subseteq f^{-1}(f(G)) \subseteq f^{-1}(f(U)) \Rightarrow f^{-1}(f(U))$ is open set in X. This completes the proof for (i) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (iv): Suppose f is β-L-continuous.

Let A be a α-closed subset of X. Then cl(int(cl(A))) is β-open set in X. By the β-L-continuity of f, we see that $f^{-1}(f(cl(int(cl(A)))))$ is open in X.

$\Rightarrow f^{-1}(f(cl(int(cl(A))))) \subseteq int(f^{-1}(f(cl(int(cl(A))))))$.

since A is α-closed in X,

We have $f^{-1}(f(cl(int(cl(A))))) \subseteq f^{-1}(f(A))$,

$\Rightarrow int(f^{-1}(f(cl(int(cl(A)))))) \subseteq int(f^{-1}(f(A)))$,

It follows that $f^{-1}(f(cl(int(cl(A))))) \subseteq int(f^{-1}(f(A)))$.

This proves (iv).

Conversely, We assume that (iv) holds.

Let U be β-open set in X $\Rightarrow f^{-1}(f(U)) \subseteq f^{-1}(f(cl(int(cl(U)))))$,

since U is α-closed by applying (iv)

we get $f^{-1}(f(cl(int(cl(U))))) \subseteq int(f^{-1}(f(U)))$,

Therefore $f^{-1}(f(U)) \subseteq int(f^{-1}(f(U)))$ and hence $f^{-1}(f(U))$ is open in X.

This proves that f is β-L-continuous.

(ii) $\Leftrightarrow$ (v): Suppose (ii) holds. Let A be a α-open subset of X. By using (ii), $f^{-1}(f^{\#}(int(cl(int(A)))))$ is closed in X

$\Rightarrow cl(f^{-1}(f^{\#}(int(cl(int(A)))))) = f^{-1}(f^{\#}(int(cl(int(A)))))$.

Since A is α-open $\Rightarrow f^{-1}(f(^{\#}(A))) \subseteq f^{-1}(f^{\#}(int(cl(int(A)))))$,

$\Rightarrow cl(f^{-1}(f(^{\#}(A)))) \subseteq cl(f^{-1}(f^{\#}(int(cl(int(A))))))$

it follows that $cl(f^{-1}(f(^{\#}(A)))) \subseteq f^{-1}(f^{\#}(int(cl(int(A)))))$,

This proves (v),

Conversely, let us assume that (v) holds.

Let A be a β-closed subset of X

$\Rightarrow f^{-1}(f^{\#}(int(cl(int(A))))) \subseteq f^{-1}(f^{\#}(A))$,

since A is α-open by (v),

we see that $cl(f^{-1}(f^{\#}(A))) \subseteq f^{-1}(f^{\#}(int(cl(int(A)))))$,

$cl(f^{-1}(f^{\#}(A))) \subseteq f^{-1}(f^{\#}(A))$, Therefore $f^{-1}(f^{\#}(A))$ is closed in X.

This proves (ii).

Theorem: 5.6

Let $f: (X,\tau) \to Y$ be a function. Then the following are equivalent.

(i) f is β-M-continuous,

(ii) for every β-open subset G of X, $f^{-1}(f^{\#}(G))$ is open in X,

(iii) $cl(f^{-1}(f(A))) \subseteq f^{-1}(f(int(cl(int(A)))))$ for every α-open subset A of X .

(iv) $f^{-1}(f^{\#}(cl(int(cl(A))))) \subseteq int(f^{-1}(f^{\#}(A)))$ for every α-closed subset A of X.

Proof:

(i) $\Leftrightarrow$ (ii): follows from theorem 5.2.

(i) $\Leftrightarrow$ (iii): Suppose f is β-M-continuous.

Let A be a α-open set in X. int(cl(int(A))) is β-closed in X, Since f is β-M-continuous, $f^{-1}(f(int(cl(int(A)))))$ is closed in X, $\Rightarrow cl(f^{-1}(f(int(cl(int(A)))))) = f^{-1}(f(int(cl(int(A)))))$

Since A is α-open in X

we see that $f^{-1}(f(A)) \subseteq f^{-1}(f(int(cl(int(A)))))$,

$\Rightarrow cl(f^{-1}(f(A))) \subseteq cl(f^{-1}(f(int(cl(int(A))))))$

$= f^{-1}(f(int(cl(int(A)))))$.

$\Rightarrow cl(f^{-1}(f(A))) \subseteq f^{-1}(f(int(cl(int(A)))))$. This proves (iii).

Conversely, suppose (iii) holds.

Let A be β-closed subset in X

$\Rightarrow f^{-1}(f(int(cl(int(A))))) \subseteq f^{-1}(f(A))$,

Since A is α-open by applying (iii),

$cl(f^{-1}(f(A))) \subseteq f^{-1}(f(int(cl(int(A)))))$,

$\Rightarrow cl(f^{-1}(f(A))) \subseteq f^{-1}(f(A))$ That implies $f^{-1}(f(A))$ is closed set in X. This completes the proof for (i) $\Leftrightarrow$ (iii).

(ii) $\Leftrightarrow$ (iv): Suppose (ii) holds. Let A be a α-closed subset of X. Then cl(int(cl(A))) is β-open in X.

By (ii), $f^{-1}(f^{\#}(cl(int(cl(A)))))$ is open in X,

$\Rightarrow f^{-1}(f^{\#}(cl(int(cl(A))))) \subseteq int(f^{-1}(f^{\#}(cl(int(cl(A))))))$,

Since A is α-closed $\Rightarrow f^{-1}(f^{\#}(cl(int(cl(A))))) \subseteq f^{-1}(f^{\#}(A))$,

$\Rightarrow int(f^{-1}(f^{\#}(cl(int(cl(A)))))) \subseteq int(f^{-1}(f^{\#}(A)))$,

we see that $f^{-1}(f^{\#}(cl(int(cl(A))))) \subseteq int(f^{-1}(f^{\#}(A)))$.

This proves (iv).

Conversely, suppose (iv) holds. Let G be β-open in X,

$\Rightarrow f^{-1}(f^{\#}(G)) \subseteq f^{-1}(f^{\#}(cl(int(cl(G)))))$.

Since G is α-closed in X, by using (iv)

$\Rightarrow f^{-1}(f^{\#}(cl(int(cl(G))))) \subseteq int(f^{-1}(f^{\#}(G)))$.

$\Rightarrow f^{-1}(f^{\#}(G)) \subseteq f^{-1}(f^{\#}(cl(int(cl(G))))) \subseteq int(f^{-1}(f^{\#}(G)))$,

$\Rightarrow f^{-1}(f^{\#}(G)) \subseteq int(f^{-1}(f^{\#}(G)))$,

Then it follows that $f^{-1}(f^{\#}(G))$ is open in X. This proves (ii).

Theorem: 5.7

Let $f: X \to (Y, \sigma)$ be a function and σ be a space with a base consisting of f^{-1}saturated open sets. Then the following are equivalent.

(i) f is β-R-continuous,

(ii) for every β-closed subset B of X, $f^{\#}(f^{-1}(B))$ is closed in Y,

(iii)for every $x \in X$ and for every β-open set V in Y with $x \in f^{-1}(V)$ there is an open set G in Y with $f(x) \in G$ and $f^{-1}(G) \subseteq f^{-1}(V)$,

(iv) $f(f^{-1}(cl(int(cl(B))))) \subseteq int(f(f^{-1}(B)))$ for every α-closed subset B of Y.

(v) $cl(f^{\#}(f^{-1}(B))) \subseteq f^{\#}(f^{-1}(int(cl(int(B)))))$ for every α-open subset B of Y.

Proof:

(i) $\Leftrightarrow$ (ii): follows from theorem 5.3.

(i) $\Leftrightarrow$ (iii): Suppose f is β-R-continuous. Let V be a β-open set in Y such that $x \in f^{-1}(V)$.
Since f is β-R-continuous, $f(f^{-1}(V))$ is open in Y.
$f(x) \in f(f^{-1}(V))$ there is an open set G in Y such that $f(x) \in G \subseteq f(f^{-1}(V))$.
That implies $x \in f^{-1}(G) \subseteq f^{-1}(f(f^{-1}(V))) \in f^{-1}(V)$. This proves (iii).
Conversely, suppose (iii) holds.
Let V be β-open in Y and $y \in f(f^{-1}(G))$, Then y=f(x) for some $x \in f^{-1}(V)$.
By using (iii) there is an open set G in Y containing f(x) such that $f^{-1}(G) \subseteq f^{-1}(V)$. We choose G to a f^{-1}–saturated in Y.
Then $G = f(f^{-1}(G)) \subseteq f(f^{-1}(V))$.
This proves that $f(f^{-1}(V))$ is open in Y. This proves that f is β-R-continuous.
(i) $\Leftrightarrow$ (iv): Suppose f is β-R-continuous. Let B be α-closed subset in Y.
Then cl(int(cl(B))) is β-open set in Y. By the β-R-continuity of f, $\Rightarrow f(f^{-1}(cl(int(cl(B)))))$ is open in Y
$\Rightarrow f(f^{-1}(cl(int(cl(B))))) \subseteq int(f(f^{-1}(cl(int(cl(B))))))$.
Since B is α-closed in Y $\Rightarrow f(f^{-1}(cl(int(cl(B))))) \subseteq f(f^{-1}(B))$,
$\Rightarrow int(f(f^{-1}(cl(int(cl(B)))))) \subseteq int(f(f^{-1}(B)))$.
Then It follows that $f(f^{-1}(cl(int(cl(B))))) \subseteq int(f(f^{-1}(B)))$.
This proves (iv).
Conversely, we assume that (iv) holds.
Let B be β-open set in Y $\Rightarrow f(f^{-1}(B)) \subseteq f(f^{-1}(cl(int(cl(B)))))$.
Since B is α-closed by applying (iv),
we get $f(f^{-1}(cl(int(cl(B))))) \subseteq int(f(f^{-1}(B)))$
Therefore $f(f^{-1}(B)) \subseteq int(f(f^{-1}(B)))$ and hence $f(f^{-1}(B))$ is open in Y. This proves that f is β-R-continuous.
(ii) $\Leftrightarrow$ (v): Suppose (ii) holds.
Let B be a α-open subset of Y.
By using (ii) $f^{\#}(f^{-1}(int(cl(int(B)))))$ is closed in Y.
$\Rightarrow cl(f^{\#}(f^{-1}(int(cl(int(B)))))) = f^{\#}(f^{-1}(int(cl(int(B)))))$
Since B is α-open in Y,
we see that, $f^{\#}(f^{-1}(B)) \subseteq f^{\#}(f^{-1}(int(cl(int(B)))))$,
$\Rightarrow cl(f^{\#}(f^{-1}(B))) \subseteq cl(f^{\#}(f^{-1}(int(cl(int(B))))))$,
it follows that $cl(f^{\#}(f^{-1}(B))) \subseteq f^{\#}(f^{-1}(int(cl(int(B)))))$.
This proves (v).
Conversely, let us assume that (v) holds.
Let B be a β-closed subset of Y
$\Rightarrow f^{\#}(f^{-1}(int(cl(int(B))))) \subseteq f^{\#}(f^{-1}(B))$,
since B is α-open in Y, by (v),
we see that $cl(f^{\#}(f^{-1}(B))) \subseteq f^{\#}(f^{-1}(int(cl(int(B)))))$,
$\Rightarrow cl(f^{\#}(f^{-1}(B))) \subseteq f^{\#}(f^{-1}(int(cl(int(B))))) \subseteq f^{\#}(f^{-1}(B))$
$\Rightarrow cl(f^{\#}(f^{-1}(B))) \subseteq f^{\#}(f^{-1}(B))$, Therefore $f^{\#}(f^{-1}(B))$ is closed in Y.
This proves (ii).

Theorem: 5.8
Let f: X $\rightarrow$ (Y, σ) be a function. Then the following are equivalent.
(i) f is β-S-continuous,
(ii) for every β-open subset V of Y, $f^{\#}(f^{-1}(V))$ is open in Y,
(iii) $cl(f(f^{-1}(B))) \subseteq f(f^{-1}(int(cl(int(B)))))$ for every α-open subset B of Y.
(iv) $f^{\#}(f^{-1}(cl(int(cl(B))))) \subseteq int(f^{\#}(f^{-1}(B)))$ for every α-closed subset B of Y.

Proof:
(i) $\Leftrightarrow$ (ii): follows from theorem 5.4.
(i) $\Leftrightarrow$ (iii) :Suppose f is β-S-continuous. Let B be a α-open set in Y.
Since f is β-S-continuous, $f(f^{-1}(int(cl(int(B)))))$ is closed in Y,
$\Rightarrow cl(f(f^{-1}(int(cl(int(B)))))) \subseteq f(f^{-1}(int(cl(int(B)))))$.
Since B is α-open in Y,
we see that $f(f^{-1}(B)) \subseteq f(f^{-1}(int(cl(int(B)))))$,
$\Rightarrow cl(f(f^{-1}(B))) \subseteq cl(f(f^{-1}(int(cl(int(B))))))$,
$\Rightarrow cl(f(f^{-1}(B))) \subseteq cl(f(f^{-1}(int(cl(int(B))))))$
$\subseteq f(f^{-1}(int(cl(int(B)))))$. This proves (iii).
Conversely, suppose (iii) holds.
Let B be β-closed subset in Y
$\Rightarrow f(f^{-1}(int(cl(int(B))))) \subseteq f(f^{-1}(B))$
Since B is α-open by applying (iii)
$\Rightarrow cl(f(f^{-1}(B))) \subseteq f(f^{-1}(int(cl(int(B)))))$,
$\Rightarrow cl(f(f^{-1}(B))) \subseteq f(f^{-1}(int(cl(int(B))))) \subseteq f(f^{-1}(B))$
$\Rightarrow cl(f(f^{-1}(B))) \subseteq f(f^{-1}(B))$,
That implies $f(f^{-1}(B))$ is closed set in Y. This completes the proof for (i) $\Leftrightarrow$ (iii).
(ii) $\Leftrightarrow$ (iv): Suppose (ii) holds. Let B be a α-closed subset of Y. Then cl(int(cl(B))) is β-open in Y.
By (ii), $f^{\#}(f^{-1}(cl(int(cl(B)))))$ is open in Y,
$\Rightarrow f^{\#}(f^{-1}(cl(int(cl(B))))) \subseteq int(f^{\#}(f^{-1}(cl(int(cl(B))))))$.
Since B is a α-closed, it follows that $f^{\#}(f^{-1}(cl(int(cl(B))))) \subseteq f^{\#}(f^{-1}(B))$
$\Rightarrow int(f^{\#}(f^{-1}(cl(int(cl(B)))))) \subseteq int(f^{\#}(f^{-1}(B)))$,
$\Rightarrow f^{\#}(f^{-1}(cl(int(cl(B))))) \subseteq int(f^{\#}(f^{-1}(cl(int(cl(B)))))) \subseteq int(f^{\#}(f^{-1}(B)))$,
we see that $f^{\#}(f^{-1}(cl(int(cl(B))))) \subseteq int(f^{\#}(f^{-1}(B)))$.
This proves (iv).
Conversely, suppose (iv) holds.
Let V be β-open in Y
$\Rightarrow f^{\#}(f^{-1}(V)) \subseteq f^{\#}(f^{-1}(cl(int(cl(V)))))$.
Since V is α-closed in Y, by using (iv), $f^{\#}(f^{-1}(cl(int(cl(V))))) \subseteq int(f^{\#}(f^{-1}(V)))$,
$\Rightarrow f^{\#}(f^{-1}(V)) \subseteq f^{\#}(f^{-1}(cl(int(cl(V))))) \subseteq int(f^{\#}(f^{-1}(V)))$,
$\Rightarrow f^{\#}(f^{-1}(V)) \subseteq int(f^{\#}(f^{-1}(V)))$,
Then it follows that $f^{\#}(f^{-1}(V))$ is open in Y. This proves (ii).

6. CONCLUSION:

In this paper the notions of β-L-Continuity, β-M-Continuity, β-R-Continuity and β-S-Continuity of a function f: X$\rightarrow$Y between a topological space and a non empty set are introduced. The purpose of this paper is to introduce, β-ρ-continuity. Here we discuss their links with β-open, β-closed sets. Also we establish pasting lemmas for β-R-continuous and β-S-continuous functions and obtain some characterizations for, β-ρ-continuity. We have put forward some examples to illustrate our notions

REFERENCES

[1] Levine N., Semi-Open sets and Semi-Continuity in Topological Spaces, Amer. *Math. Monthly*, 70 (1963), 36-41.

[2] Mashhour A.S, Abdel Monsef M.E and E.I Deeb S.N., On Pre-Continuous and Weak Pre-Continuous mappings, Proc. *Math. Phys. Soc. Egypt* 53 (1982), 47-53.

[3] Navpreet singh Noorie and Rajni Bala, Some Characterizations of open, closed and continuous Mappings, *International J. Math. & Math. Sci.*, Article ID 527106, (2008),1-5.

[4] Selvi R.,Thangavelu P.,and Anitha M., ρ-Continuity Between a Topological space and a Non Empty Set, where $\rho \in$ {L, M, R, S}, *International Journal of Mathematical Sciences*,9(1-2)(2010), 97-104.

[5] Stone M. H., Applications of Theory of Boolean Rings to General Topology, Trans. A.M.S, 41(3),(1937), 375-481.

[6] Thangavelu P., Selvi R., On Characterization of ρ-Continuity where $\rho \in$ {L, M, R, S}, *International journal of applied mathematical analysis and applications*, January – June 2012, Volume 7, page no 153-159.

[7] Njastad O., On some classes of nearly open sets, *pacific.j.Math.*, 15(1965),961-970.

[8] Andrijevic D., semi-pre-open sets, *Mat. Vesnik*, 38(1)(1986), 24-32.

[9] priyadarshini M., Thangavelu P., Selvi R., On pre-ρ-Continuity where $\rho \in$ {L, M, R, S}, *International journal of science and Research*, Volume 4, Issue 4, April 2015, page no 278-283.

[10] priyadarshini M., Selvi R., On Semi-ρ-Continuity where $\rho \in$ {L, M, R, S}, *International journal of Technical Research and applications*, volume 3, Issue 3 , May 2015, page no: 221-226